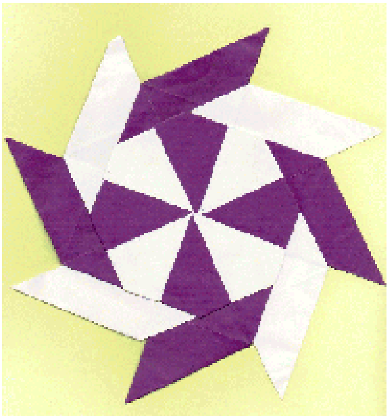
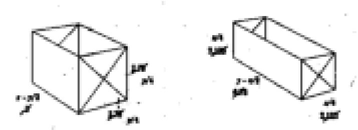
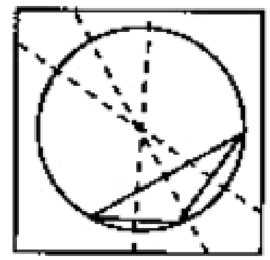
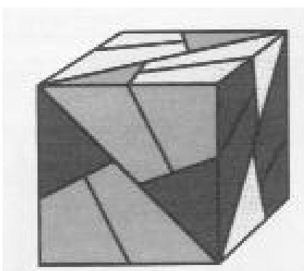
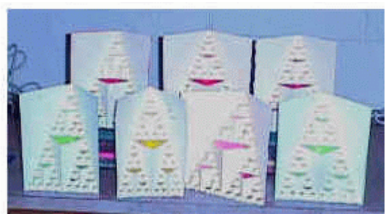
Paper-Folding Ideas to Help Students Understand High School Geometry Concepts

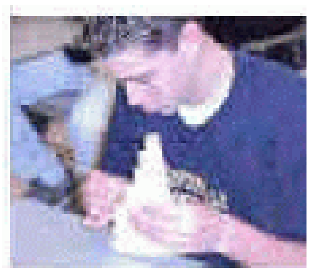












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Paper Folding an Octagon

Activity: To construct an octagon with 8 pieces of unit origami.

Materials: Use two colors, four pieces of each color.

Steps:

Step 1: Begin with a square piece of paper.

Step 2: Assume that the origin of an x, y coordinate system is in the center of the square.

Fold in half on the y axis. Open back to a square.
Fold in half on the line y = x. Open back to a square.
Fold in half on the line y = -x. Open back to a square.
Identify the fractional parts of the square bounded by the folds.

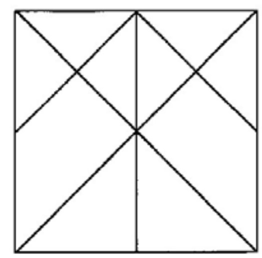

Figure 7

See Figure 1

Step 3: Fold the “maximum point”, y = x to the origin.
Describe the triangle formed by the fold.
Fold the “maximum point”, y = -x to the origin.
Describe the triangle in terms of the previous triangle

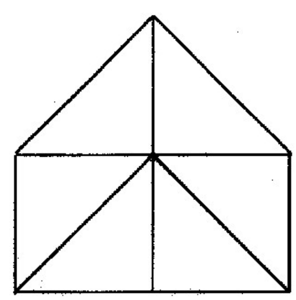

Figure 8

See Figure 2

Step 4: Fold in half on the y axis.
Fold (0, -y) inside to meet point (x, 0). Describe the figure. What fractional part of the original square is its area? (See Figure 3)

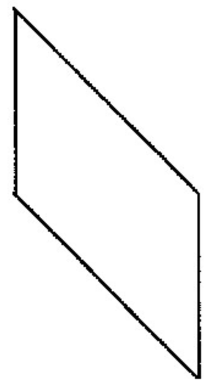

Figure 9

Step 5: Place two parallelogram units in the same position, so that the close end is at the bottom right. There should then be an open end at the top left.
(Figure 4)

Figure 10

Step 6: Slide the top parallelogram into the open part of the bottom parallelogram, rotate the top parallelogram 45 degrees until the long side lies along the inside fold. (Figure 5)

Figure 11

Two small isosceles right triangles should appear above the inside parallelogram. (Figure 6)

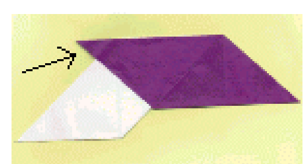

Figure 12

Fold these two isosceles right triangles inside the parallelogram.(Figure 7)

Figure 13

Repeat this step with the other six parallelograms until you have connected all eight parallelograms.(Figure 8)

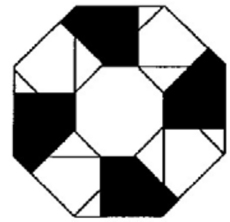

Figure 14

Follow-up Activities with the Octagon:

If the edge length of the interior octagon is one, what is the area of the interior octagon?

Trace the interior octagon onto a piece of 1/4" graph paper. Determine the number of squares in the interior by counting.

Use the Pythagoreum Theorem to help you determine the area of the Octagon. How do the counting and mathematical methods compare?

Repeat for the exterior octagon. Include the interior octagon as part of the area of the exterior octagon. What is the edge length of the exterior octagon assuming that the interior octagon edge is 1.

What is the ratio of the edge lengths of the two octagons? What is the ratio of the areas of the two octagons?

What is the square of the ratio of the edge lengths? How does this value compare to the ratio of the areas?

Slide the octagon together to form a star. (Figure 9)

Figure 15

Into what shapes can you make the hole at the center of star?

Creating an Open Box Using Origami

Activity: To create two different open boxes using paper-folding.

Materials: Two sheets of 8.5" x 11" paper.

Steps:

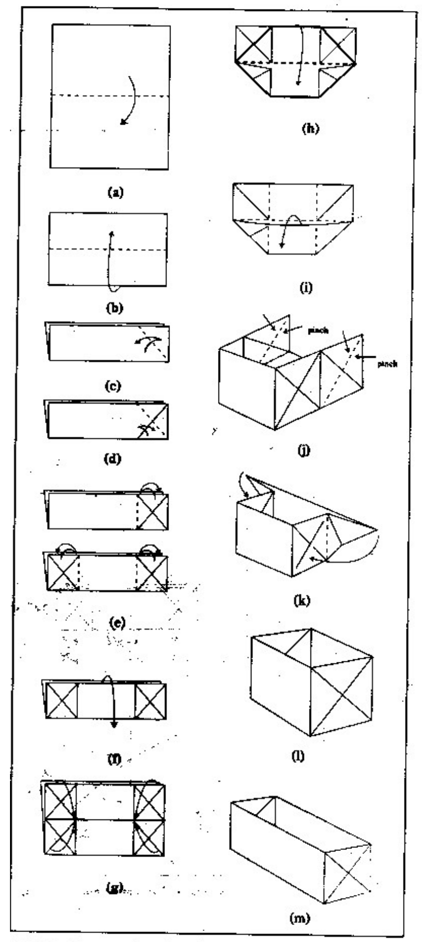

Fig. 1 Creating an open box using origami

Step a: Fold paper in half so new paper is 8.5' x 5.5". Leave folded. Fold should be at the top.

Step b: Fold the paper again in half by bringing the open side up to the fold. Leave folded.

Step c: Fold the top right hand corner down to form an isosceles right triangle. Open this fold.

Step d: Fold the bottom right hand corner up to form another isosceles right triangle. Open the fold.

Step e: Fold over the right side to form an X inside a square. Open the fold. Repeat steps b, c, and d with the left hand side.

Step f: Open the rectangle. Turn the rectangle until the open side is at the top.

Step g: Fold one layer of the top left and right hand corners down to form an isosceles right triangle. Fold both layers from the bottom right and left hand corners up to form two other isosceles right triangles.

Step h: Fold down the top from trapezoid to match the trapezoid at the bottom.

Step i: This last fold forms a pocket. Reach in and pull open the box. Form the edges of the box.

Step j: Reverse fold the diagonal and then fold backwards the isosceles triangle to fit into the already formed pocket. Repeat this for both sides.

You have formed an open box. What if you fold the paper the other way in step a.

Repeat steps a-j with the paper held the other direction.

Activities with the two boxes:

How does the volume of the one box compare to the other?

What are some of the ways the two volumes can be compared?

Can you compare the two volumes algebraically?

If one sheet of normal paper is cut in half, how will the volume of the original box compare with the new box?

If the sheet of normal paper is subdivided in fourths, how will the volume of the original box compare to the new box?

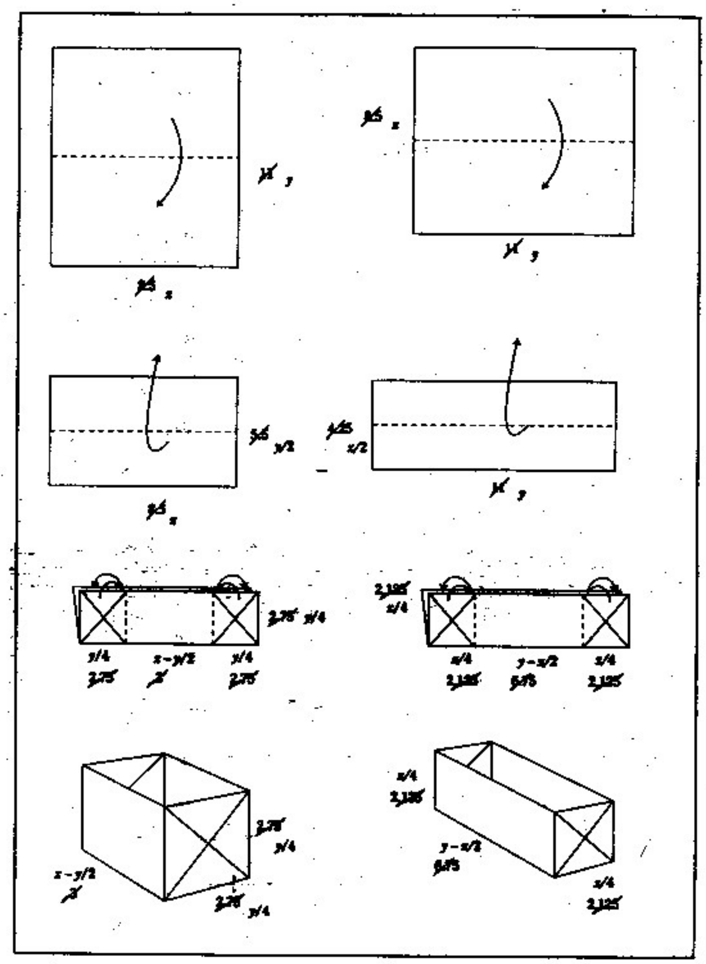

Fig. 4 Using variables in the folding sequence

What will the box look like if you use normal origami paper to make the box? What shape will it be?

Activity from:
Algebraic Thinking through Origami

Discovering Properties of Triangles with Paper Folding

Activity: To fold the perpendicular bisectors, angle bisectors, and medians of a triangle.

Materials: Patty Paper, ruler, and pencil

Steps:

Step 1: Draw a large acute scalene triangle on your patty paper with a pencil.

Step 2: Make three copies of your triangle.

Step 3: Fold the patty paper to construct the perpendicular bisectors of each side of your triangle.

Step 4: What do you notice about these three perpendicular bisectors? Mark this point of concurrency as the circumcenter, using the letter C.

Place a second patty paper over the triangle. Mark the distance from the circumcenter to one of the vertices. Compare this distance with the distance to the other two vertices. How do they compare?

Step 5: Take another copy of your large acute scalene triangle and fold the three angle bisectors.

Step 6: What do you notice about these three angle bisectors? Mark this point of concurrency as the incenter, using the letter I.

Slide the edge of a second patty paper along one side of the acute triangle until an adjacent perpendicular side of the patty paper passes through the triangle's incenter. Mark this distance on the patty paper. Compare this marked distance with the distance to the other two sides.

Step 7: Take another copy of your large acute scalene triangle and fold the three medians of the triangle.

Step 8: What do you notice about these three medians? Mark this point of concurrency as the centroid or center of mass, using the letter M.

Step 9: Take another copy of your large acute scalene triangle and fold the three altitudes.

Step 10: What do you notice about these three altitudes? Mark this point of concurrency as the orthocenter, using the letter O.

Step 11: Place the four triangles on top of each other and notice if points C, I, M, and O are all identical.

Step 12: Repeat steps 1-11 using a large obtuse scalene triangle. Place the obtuse trangle so it only takes up about ½ of the paper diagonally. Place the longest side of the triangle toward the middle of the paper.

Step 13: Repeat steps 1-11 using a large right scalene triangle. To create a right triangle, first fold a right angle, draw two sides on this right angle and then add the hypotenuse.

Step 14: Repeat steps 1-11 using a large isosceles triangle.

Step 15: Repeat steps 1-11 using a large equilateral triangle.

Activities:

What can you describe about the position of the circumcenter of a triangle?

What can you describe about the position of the incenter of a triangle?

What can you describe about the position of a centroid of a triangle?

Do any of the four points line up to form a line? Which points line up to form Euler's line?

Unit Origami with a Box

Unit Origami with a Box

Activity: To create a unit origami box using six unit pieces. Compare surface area and volume of boxes that have dimensions that are twice each other.

Materials: 6 sheets of origami paper (3 colors, 2 of each)

In this activity you will learn how to fold a traditional origami unit that can be used to create many different polyhedra. The folding is easy and will become so mechanical that you will be able to do it without thinking. David Masunaga has explored many different polyhedra that can be made from the star-building unit. You will be exploring some of these polyhedra in the activities that follow.

MATERIALS NEEDED FOR EACH STUDENT

One or two squares of origami or patty paper
Origami journal

GROUPING

Work with a partner in your group. Each student should assemble his or her own star-building unit.

REMINDER

Save your completed units for the next activity.

FOLDING INSTRUCTIONS AND QUESTIONS

When you are folding, think about the answers to the folding questions asked at each step. When you have finished folding, answer these questions in your origami journal.

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1 Start with the white side of the paper facing up. Fold the paper into two congruent rectangles and unfold.

What can you say about the area of each small rectangle compared to the area of the square?

2 Fold each small rectangle in half lengthwise and unfold.

a. *What is the area of each of the smallest rectangles compared to the area of the square?*

b. *Describe the relationship among the three folded lines on your paper.*

3 Fold the lower corner up.

a. *What kind of triangle have you folded?*

b. *What are the measures of its angles?*

4 Rotate the paper 180° and repeat step 3.

a. *What is the name of the six-sided polygon you have made? Is it a regular polygon? Why or why not?*

b. *Put your finger on the center point of the paper and rotate the figure 180°. Explain why you can say that this figure has 180° rotational symmetry.*

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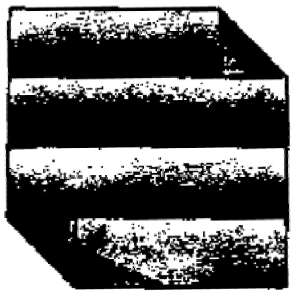
5 Fold to bisect the 45° angle as shown. This fold is known as the "paper airplane" fold. Be sure to keep the vertex point as sharp as possible.

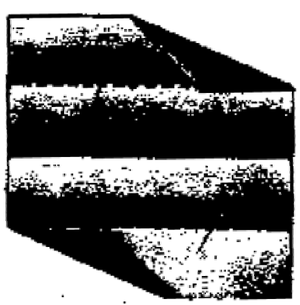


6 Rotate the paper 180° and repeat step 5.

a. *What kind of triangle is this final folded triangle?*

b. *What are the measures of its angles?*

7 Refold along the existing parallel line segments *AB* and *CD*.

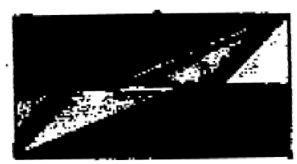


8 Starting from the lower right-hand corner, fold a large isosceles triangle so that point *F* lies on point *G* and point *H* is a vertex.

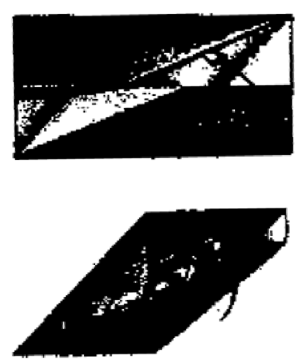
a. *Look for the two congruent triangles in the diagram above. What kind of triangles are these?*

b. *What kind of polygons are the two congruent patterned figures?*

9 Rotate the paper 180° and repeat step 8.



10 Tuck each flap into a pocket, making sure the corners lie flat when inserted.



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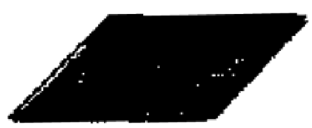
The star-building unit is now complete. Your figure should now look like the one below. The size of the white space on your unit, as shown in the picture below, reveals how accurate your folds have been. A small white space won't make much difference, but a gaping hole might.

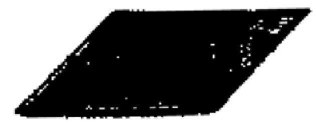
Here's how to fold the unit for easy storage. Turn the figure over.



a. *Name the quadrilateral formed.*

b. *What are the measures of the angles of this quadrilateral?*

13 Fold the top right acute angle so that its vertex lies on top of the top left obtuse angle.



a. *What kind of triangle is formed by your fold?*

b. *What can you say about the area of the triangle compared to the area of the quadrilateral?*

14 Rotate the figure 180° and repeat step 13.

15 Leave the unit in its square shape, and put it in a safe place.

How does the area of the final square compare to the area of the quadrilateral in step 12?

Remember to answer each of the folding questions in your origami journal and save your completed star-building units for the next activity.

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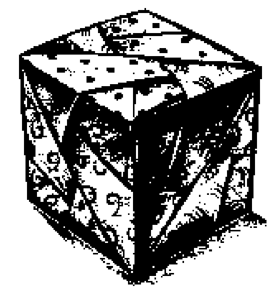
Cube



In this activity you will make a cube using six star-building units. As you assemble your cube, try to visualize what the final product will look like. This is easy to do with a cube because it is probably the best-known polyhedron. Also notice the pattern forming on each face of the cube as you tuck the tabs into the pockets.

MATERIALS NEEDED FOR EACH STUDENT

Six 6-inch squares of origami paper—two each of three different colors
Triangular hexahedron from the previous activity
Six 3-inch squares of origami paper—two each of three different colors
Rubber bands
Skewer
Origami journal

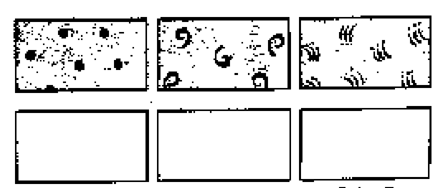
GROUPING

Work with a partner. Each person should assemble his or her own cube.

ASSEMBLY INSTRUCTIONS AND QUESTIONS

When you are assembling the cube, think about the answers to the questions asked at each step. When you have finished assembling your model, answer these questions in your origami journal.

Before you start assembling the cube, fold six star-building units, two each of three different colors. Record the colors you used in the rectangles at the right.



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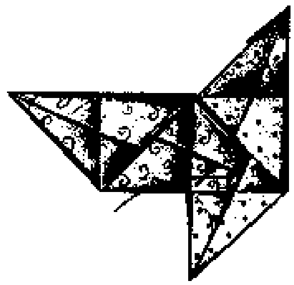
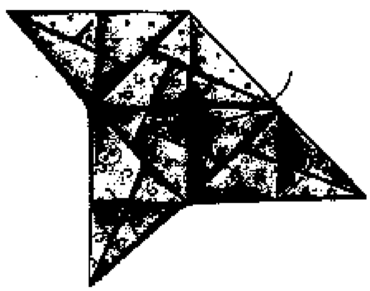
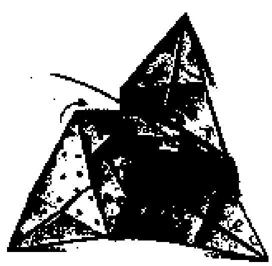
1 Fit a tab of a color 2 unit into a pocket of a color 1 unit as shown.



2 Rotate the figure 90° counterclockwise. Next, fit a tab of the color 3 unit into the pocket of the color 2 unit as shown.

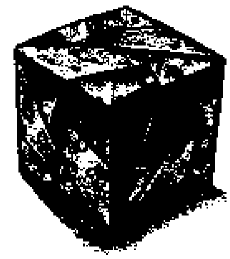


3 Form a pyramid by inserting the color 1 unit into the color 3 unit. You have now formed one corner of the cube. Notice that the pockets are on the outside of the cube.



What kind of pyramid is formed in this step?

4 Complete the cube with the three remaining star-building units. The following picture shows three faces of the completed cube. Make sure that each face has only two colors on it.



Each square face is composed of only two colors. Opposite faces have the same colors. How can these facts help you decide what color to add to complete each face?

5 You have made a cube. Notice that there is a one-to-one correspondence between tabs and pockets. There are no stray tabs and no empty pockets. If you have empty pockets, check inside the cube to see if there are any stray tabs.

Remember to answer each of the assembly questions in your origami journal.

EXPLORING YOUR MODEL

Record your answers to the exploring questions in your origami journal.

1. Use rubber bands to outline the reflection planes of the cube. Don't forget that a plane can cut a cube diagonally. Draw a sketch to show each reflection plane.
2. Use skewers to find the axes of rotational symmetry of the cube (you may have to puncture the paper with the skewers). How many axes of 4-fold rotational symmetry are there? How many axes of 3-fold rotational symmetry are there? How many axes of 2-fold rotational symmetry are there? (For the axes of 2-fold symmetry, it may be difficult to use skewers. Rotate the cube between two fingers instead.)
3. Describe the rotational symmetries of the cube using degrees.
4. Assemble a second cube. This time, use square papers whose sides are half the length of your original paper. Copy the chart below into your origami journal. Make predictions in the bottom row of the chart. Then fill in the rest of the chart. How accurate were your predictions?

COMPARING TWO CUBES

	Edge length of paper	Edge length of cube	Surface area of cube	Volume of cube
Original cube				
Smaller cube				
Ratio: $\frac{\text{Original cube}}{\text{Smaller cube}}$				

5. Find the exact length of a side of the cube. (Hint: Unfold one star-building unit, look at the fold lines, and find the outline of one of the cube faces in the fold lines.)
6. Look at the colors on your cube. Describe one of the patterns that you see.

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7. How many different-looking cubes can you make from your six star-building units? Explain your reasoning.

8. Try building different-sized cubes. (You might want to see who can build the smallest one and who can build the largest one.) Create a chart similar to the one below. Collect results from your group or your class.

Name of student	Edge length of paper (in.)	Edge length of cube (in.)	Surface area of cube (sq in.)	Volume of cube (cu in.)

9. Make three separate graphs to show the relationship between the edge length of the paper and the edge length of the cube, the edge length of the paper and the surface area of the cube, and the edge length of the paper and the volume of the cube. Describe how the graphs differ.

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Fractal Pop-Up Cards

Activity: To create three-dimensional pop-up cards and then study patterns created through the construction.

Materials: Two pattern sheets
Two backing sheets
Glue Sticks

Steps for Card 1:

Step 1: Fold the pattern sheet in half so the one set of line segments ends on the fold. Notice there are 4 different length segments on the sheet. You will be cutting along the longest segment each time and then completing a reverse fold.

Step 2: Cut along the two longest line segments . Fold the center rectangle at the end points of the next longest line segments.

Step 3: Open the card and reverse fold the rectangle inside.

Step 4: Repeat steps 2 and 3 for the next longest line segment. You have two rectangles that will need to be folded inside on this step.

Step 5: Repeat steps 2 and 3 for the next longest line segment. You will have four rectangles that will need to be folded inside on this step. Because of the thickness of the paper you want to fold two rectangles at a time. Then complete the reverse folding.

Step 6: Repeat steps 2 and 3 for the next longest line segment. You will have eight rectangles that will need to be folded inside on this step. Because of the thickness of the paper you want to fold two rectangles at a time. Then complete the reverse folding.

Step 7: Glue a sheet of colored paper that is already folded in half to the back of the card. It is best to glue one side at a time. Once you have glued both sides, you can open the card to see a pop-up design.

Activity Follow-up Questions:

Find the pattern created by this pop up card design.

Stage	1	2	3	4	5	6	n
Number of boxes							
Total Number of all size boxes							

How are total number of all size boxes related to the Tower of Hanoi?

Steps to Card 2:

Step 1: Fold the pattern sheet in half so that three segments start on the fold. Notice there are three different length segments.

Step 2: Cut along the longest line segment that is in the center of the arrangement of line segments.

Step 3: Fold the left side at the base of the other line segments. Open the card and reverse fold the rectangle inside the card.

Step 4: Repeat steps 2 and 3 by cutting along the second longest lines segment. You should have two middle sized line segments. One cut should be through four layers of paper and one cut should be through two layers of paper. This should create two places you can fold so the fold is at the base of shortest line segments. The rectangles should be on the left of each cut you just made. After you fold the rectangle, remember to reverse fold them inside the card.

Step 5: Repeat steps 2 and 3 by cutting along the shortest lines segment. You should have four shortest sized line segments. Each cut will be through a different number of layers. Fold all the rectangles on the left of the cut you just made and then reverse fold them.

Step 6: Glue a sheet of colored paper that is already folded in half to the back of the card. It is best to glue one side at a time. Once you have glued both sides, you can open the card to see a pop-up design.

Step 7: (Optional) If you want to create another stage you can cut each step in half and cut up halfway up the step and fold and reverse fold each rectangle. There will be a lot of rectangles, be patient.

Follow Up Activity:

Find the pattern created by this pop up card design.

Stage	1	2	3	4	5	6	n
Number of boxes	1						

How are these numbers related to the number of triangles in Sierpinski's Triangle?